Pergamon

PII: S0021-8928(98)00098-7

J. Appl. Maths Mechs, Vol. 62, No. 5, pp. 771–776, 1998

0021-8928/98/$—see front matter

THE FICTITIOUS ABSORPTION METHOD IN DYNAMIC ELECTROELASTICITY PROBLEMS†

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(*Received* 26 *August* 1997)

The fictitious absorption method [1] is developed to solve convolution-type integral equations, specified in a system of sections and generated by dynamic problems of the excitation of a piezoelectric crystal by a system of strip electrodes. Compared with other approaches this method enables solutions to be constructed with high accuracy simultaneously in all the regions in which the integral equations are specified, including the boundary, and can be used at any frequencies.

Dynamic problems of analysing multielectrode structures by the method of integral transformations [1, 2] reduce to the convolution-type integral equations

$$\sum_{r=1}^{N} Kq_r(x) = f_m, \quad x \in \Omega_m, \quad \Omega_m = [a_{2m-1}, a_{2m}] \tag{1}$$

$$Kq_r(x) = \int_{a_{2r-1}}^{a_{2r}} k(x-\xi)q_r(\xi)d\xi, \quad k(x) = \frac{1}{2\pi}\int_\sigma K(\alpha)e^{-i\alpha x}d\alpha$$

Here and everywhere below, unless otherwise stated, the subscript m takes values of $1, 2, \ldots, N$ (N is the number of electrodes on the surface of the piezo-electric crystal).

The functions q_m and f_m have carriers in the interval Ω_m. The contour is situated in accordance with the rules which ensure that the radiation conditions at infinity are satisfied [3]. Assuming system (1) to be uniquely solvable in L_p, where $p > 1$ for any twice continuously differentiable function f_m [2, 4], we will assume that the function $K(\alpha)$ possesses the following properties, characteristic of a wide class of dynamic mixed problems in the theory of elasticity:

1. it is an even function of the parameter α, and meromorphic in the complex plane;
2. along the real axis there can be a finite number of real zeros and poles z_k, p_k ($k = 1, 2, \ldots, N_0$) and a denumerable set of complex zeros and poles z_k, p_k ($k = N_0 + 1, \ldots, \infty$) from the point of condensation in sectors of small angles containing the imaginary axis

$$3.\ \ K(\alpha) = c|\alpha|^{-1}[1 + O(\alpha^{-1})], \quad |\alpha| \to \infty.$$

Without loss of generality we will construct solutions $q_m(x)$ of Eqs (1) with right-hand side $f_m = A_m e^{-i\eta x}$ (A_m, η = const), and assuming that the constant c, characterizing the behaviour of K at infinity, is equal to unity.

By the fictitious-absorption method the function $K(\alpha)$ can be represented in the form of a product $K(\alpha) = S(\alpha)\Pi(\alpha)$.

We will choose the function $S(\alpha) = (\alpha^2 + B^2)^{-1/2}$, $B > 0$ as $S(\alpha)$.

It is obvious that $S(\alpha)$ is a regular function on the real axis, and its asymptotic behaviour is identical with the behaviour of $K(\alpha)$ as $|\alpha| \to \infty$.

The function $\Pi(\alpha) = S^{-1}(\alpha)K(\alpha)$ can be approximated by a rational function of the form

$$\Pi(\alpha) = \prod_{k=1}^{n} \frac{\alpha^2 - z_k^2}{\alpha^2 - p_k^2}, \quad \Pi(\alpha) = 1 + O(\alpha^{-1}), \quad |\alpha| \mapsto \infty \tag{2}$$

using Bernshtein or Lagrange polynomials, as was described in detail earlier [3]; $s = n - N_0$ is the degree of the approximating polynomial. The number of zeros and poles of the function $\Pi(\alpha)$ depends on the

†*Prikl. Mat. Mekh.* Vol. 62, No. 5, pp. 834–839, 1998.

desired accuracy with which the approximate solution is constructed, where the first N_0 zeros and poles of the function are identical with the singular points of $K(\alpha)$ on the real axis. The use of this approximation was justified earlier in [1, 3].

Using the fictitious-absorption method [1] we will seek a solution in the form (everywhere henceforth $j = 1, 2, \ldots, n$)

$$q_m(x) = q_m^0(x) + \varphi_m(x), \quad \varphi_m(x) = \sum_{k=1}^{2n} c_{km}\delta(x - x_{km}) \tag{3}$$

when the following conditions are satisfied

$$\int_{a_{2m-1}}^{a_{2m}} q_m(x)e^{\pm ip_j x}dx = \sum_{k=1}^{2n} c_{km}e^{\pm ip_j x_{km}} \tag{4}$$

where p_j are the poles of the functions $\Pi(\alpha)$, situated above the contour σ, c_{km} are unknown constants, to be determined, and x_{km} are points which divide the intervals Ω_m into equal sections

$$x_{km} = a_{2m-1} + k(a_{2m} - a_{2m-1})/(2n+1)$$

Lemma 1. Suppose the functions $q_m^0(x) \in L_p(a_{2m-1}, a_{2m}), p > 1$ and have carriers in the interval Ω_m. In order that the functions

$$t_m(x) = \int \Pi(\alpha)Q_m^0(\alpha)e^{-i\alpha x}d\alpha$$

should have the same properties it is necessary and sufficient that the functions

$$Q_m^0(\alpha) = \int_{-a}^{a} q_m^0(x)e^{i\alpha x}dx$$

satisfy the conditions $Q_m^0(\pm p_j) = 0$ in the polar set $\Pi(\alpha)$.

The functions $q_m^0(x)$ satisfy the conditions of the lemma, by virtue of (4), and on the basis of this we will introduce new unknowns $t_m(x)$ by the relations

$$t_m(x) = \frac{1}{2\pi}\int_\sigma T_m(\alpha)e^{-i\alpha x}d\alpha, \quad T_m(\alpha) = \Pi(\alpha)Q_m^0(\alpha) \tag{5}$$

Introducing expressions (3) into Eqs (1) and taking relations (5) into account, we arrive at the following system of integral equations with regular kernel in $t_m(x)$

$$\sum_{r=1}^{N} St_r(x) = g_m(x), \quad x \in \Omega_m; \quad g_m = f_m - \sum_{k=1}^{2n}\sum_{r=1}^{N} c_{kr}k(x - x_{kr}) \tag{6}$$

$$St_m = \int_{a_{2m-1}}^{a_{2m}} s(x-\xi)t_m(\xi)d\xi, \quad s(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\alpha)e^{-i\alpha x}d\alpha$$

Theorem. Suppose $t_m^\eta(x)$ are solutions of the equations

$$St_m^\eta(x) = e^{-i\eta x}, \quad x \in \Omega_m \tag{7}$$

Then, the integral representation of the solution of the system of integral equations (1) for $f_m = A_m e^{-i\eta x}$ is given by the relations

$$\begin{aligned} q_m(x,\eta) = {} & A_m t_m^\eta(x) + \frac{A_m}{2\pi}\int_\sigma [\Pi^{-1}(\alpha) - 1]T_m^\eta(\alpha)e^{-i\alpha x}d\alpha - \\ & -\frac{1}{2\pi}\sum_{k=1}^{2n}\sum_{r=1}^{N} c_{kr}\left\{\int_\sigma \Pi^{-1}(\alpha)L_m(\alpha, x_{kr})e^{-i\alpha x}d\alpha + \right. \\ & \left. +\int_\sigma [\Pi^{-1}(\alpha) - 1]e^{-i\alpha(x - x_{kr})}d\alpha\right\}, \quad x \in \Omega_m \end{aligned} \tag{8}$$

where the constants c_{kr} are found from the system of equations

$$\sum_{k=1}^{2n}\sum_{r=1}^{N} c_{kr}[e^{\pm iz_j x_{kr}} + L_m(\pm z_j, x_{kr})] = A_m T_m^{\eta}(\pm z_j) \tag{9}$$

where z_j are the zeros of $\Pi(\alpha)$, which lie above the contour σ, and the functions L_m have the form

$$L_m(\alpha, x) = \frac{1}{2\pi}\int_{\sigma} S(\eta)[\Pi(\eta) - 1]T_m^{\eta}(\alpha)e^{i\eta x}d\eta \tag{10}$$

Proof. Multiply both sides of Eq. (7) by $S(\eta)[\Pi(\eta) - 1]e^{i\eta x_{kr}}$ and integrate with respect to the parameter η. Then the solution $t_m^1(x)$ of the system of equations (6) with right-hand side $k_1(x - x_{kr})$ ($k_1(x) = k(x) - s(x)$) has the form

$$t_m^1(x) = \int_{\sigma} S(\eta)[\Pi(\eta) - 1]t_m^{\eta}(x)e^{i\eta x_{kr}}d\eta$$

Using the superposition principle, and also the fact that the contribution of terms of the form $St_m(x)$ when $x \notin \Omega_m$ can be neglected compared with $St_m(x)$ when $x \in \Omega_m$ [3], we obtain the general solution of system (6) in the following form

$$t_m(x) = A_m t_m^{\eta}(x) - \varphi_m(x) - \frac{1}{2\pi}\sum_{k=1}^{2n}\sum_{r=1}^{N} c_{kr}\int_{\sigma} L_m(\alpha, x_{kr})e^{-i\alpha x}d\alpha -$$
$$-\frac{1}{2\pi}\sum_{k=1}^{2n}\sum_{\substack{r=1\\ r\neq m}}^{N} c_{kr}\int_{\sigma} L_m^0(\alpha, x_{kr})e^{-i\alpha x}d\alpha \tag{11}$$

The Fourier transformant of this solution has the form

$$T_m(\alpha) = A_m T_m^{\eta}(\alpha) - \sum_{k=1}^{2n}\left\{c_{km}e^{i\alpha x_{km}} + \sum_{r=1}^{N} c_{kr}L_m(\alpha, x_{kr}) + \right.$$
$$\left. + \sum_{\substack{r=1\\ r\neq m}}^{N} c_{kr}L_m^0(\alpha, x_{kr})\right\},\quad L_m^0(\alpha, x) = \frac{1}{2\pi}\int_{\sigma} S(\eta)T_m^{\eta}(\alpha)e^{i\eta x}d\eta \tag{12}$$

The functions $L_m(\alpha, x)$ are given by relations (10).
From condition (5) we obtain

$$q_m^0(x) = t_m(x) + \frac{1}{2\pi}\int_{\sigma}[\Pi(\alpha)^{-1} - 1]T_m(\alpha)e^{-i\alpha x}d\alpha \tag{13}$$

Substituting expressions (11) and (12) into the last equation and taking Eqs (3) and (13) into account, we obtain an integral representation of the solution of system (1) in the form (8).

Since $q_m^0(x) \in L_p(\Omega)$, $p > 1$ ($\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_N$) and have carriers in Ω_m, we obtain $T_m(\pm z_j) = 0$.

Hence, by (12) we obtain a linear algebraic system (9) of order $2nN$ unknowns c_{km}. The theorem is proved.

The solutions $t^{\eta}{}_m(x)$ and the corresponding Fourier transformations $T_m^{\eta}(\alpha)$ are constructed by the method of factorization [3] for the right-hand sides $f_m = A_m e^{-i\eta x}$ and are not given here.

The integrals $L_m(\alpha,x)$, $L_m^0(\alpha,x)$ in (10) and (12) are evaluated by residues after substituting the expressions for $T_m^{\eta}(\alpha)$ into them, taking into account the decrease in the integrands in the lower half-plane.

Substituting the expressions obtained for $L_m(\alpha,x)$, $T_m^{\eta}(\alpha)$, $t_m^{\eta}(x)$ into the integral representation of the solution (8) we obtain, after appropriate reduction, the approximate solution of system (1) in the form

$$q_m(x, \eta) = A_m\{u_m^{\eta} + K^{-1}(\eta)e^{-i\eta x}[v_m^{\eta}(x) - 1] +$$
$$+\sum_{r=1}^{n}\frac{\beta_r}{2z_r}[e^{-i\eta a_{2m}}\sqrt{B - i\eta}\,\Phi_r(\eta, a_{2m} - x) + e^{-i\eta a_{2m-1}}\sqrt{B - i\eta}\,\Phi_r(-\eta, x - a_{2m-1})]\} -$$

$$-i\sum_{j=1}^{N}\sum_{k=1}^{2n}c_{kj}\left[\frac{e^{-B(a_{2m}-x)}}{\sqrt{\pi(a_{2m}-x)}}P_s(a_{2m}-x_{kj})+\frac{e^{-B(x-a_{2m-1})}}{\sqrt{\pi(x-a_{2m-1})}}P_t(x_{kj}-a_{2m-1})+\right.$$

$$\left.+\Psi_s(a_{2m}-x_{kj},a_{2m}-x)+\Psi_t(x_{kj}-a_{2m-1},x-a_{2m-1})\right],\quad x\in\Omega_m \tag{14}$$

The unknowns c_{kj} are found from the linear algebraic system of order $2nN$

$$\sum_{k=1}^{2n}\sum_{j=1}^{N}c_{kj}[\sqrt{B+i\alpha}e^{i\alpha a_{2m}}F_s(\alpha,a_{2m}-x_{kj})+$$

$$+\sqrt{B-i\alpha}e^{i\alpha a_{2m-1}}F_t(-\alpha,x_{kj}-a_{2m-1})]=A_mT_m^{\eta}(\alpha),\quad \alpha=\pm z_r \tag{15}$$

We must put $s = t = 1$ when $j = m$, $s = 2$ and $t = 1$ when $j > m$, and $s = 1$ and $t = 2$ when $j < m$ in (14) and (15).

We have used the following notation

$$P_t(x)=\sum_{j=1}^{n}g_j^{\pm}(x),\quad g_j^{\pm}(x)=\alpha_j\frac{e^{\pm ip_jx}}{2p_j\sqrt{B\mp ip_j}}$$

$$\Psi_t(x,y)=\sum_{r=1}^{n}\sum_{j=1}^{n}\frac{\beta_r}{2z_r}g_j^{\pm}(x)\Phi_r(\mp p_j,y)$$

$$F_t(\alpha,\xi)=\sum_{j=1}^{n}(\alpha\pm p_j)^{-1}g_j^{\pm}(x),\quad t=1,2$$

($t = 1$ corresponds to the plus sign and $t = 2$ corresponds to the minus sign)

$$u_m^{\eta}(x)=\frac{\sqrt{B-i\eta}e^{-i\eta a_{2m}}}{\sqrt{\pi(a_{2m}-x)}}e^{-B(a_{2m}-x)}+\frac{\sqrt{B+i\eta}e^{-i\eta a_{2m-1}}}{\sqrt{\pi(x-a_{2m-1})}}e^{-B(x-a_{2m-1})}$$

$$v_m^{\eta}(x)=\operatorname{erf}\sqrt{(B+i\eta)(a_{2m}-x)}+\operatorname{erf}\sqrt{(B-i\eta)(x-a_{2m-1})}$$

$$\alpha_j=\prod_{k=1}^{n}(p_j^2-z_k^2)\prod_{\substack{k=1\\k\neq j}}^{n}(p_j^2-p_k^2)^{-1},\quad \beta_i=\prod_{k=1}^{n}(z_i^2-p_k^2)\prod_{\substack{k=1\\k\neq i}}^{n}(z_i^2-z_k^2)^{-1}$$

The Fourier transformants of the solutions $q_m(x, \eta)$ have the form

$$Q_m(\alpha,\eta)=\frac{1}{\Pi(\alpha)}\{A_mT_m^{\eta}(\alpha)-\sum_{k=1}^{2n}\sum_{j=1}^{N}c_{kj}[\sqrt{B+i\alpha}e^{i\alpha a_{2m}}*$$

$$*F_s(\alpha,a_{2m}-x_{kj})+\sqrt{B-i\alpha}e^{i\alpha a_{2m-1}}F_t(-\alpha,x_{kj}-a_{2m-1})]\} \tag{16}$$

Remarks. 1. The solution of integral equation (1) has been obtained assuming $c = 1$, i.e. apart from the factor c^{-1}, where c is a constant characterizing the behaviour of the function $K(\alpha)$ at infinity.

2. The physically realizable electrical boundary conditions in mixed problems of acousto-electronics have a more special form: $f(x) = \psi_m$ = const on each of the N electrodes. In this case $\eta = 0$. For contact problems of the theory of elasticity, the function $f(x)$ describes the form of the base of the punch.

3. Using the solutions $q(x, \eta)$, obtained for the right-hand side of $e^{-i\eta x}$, it is easy to construct a solution for an arbitrary right-hand side of $f(x)$, if we represent it by the integral $f(x) = \int_\sigma F(\eta)e^{-i\eta x}d\eta$. Then $q(x) = \int_\sigma q(x, \eta)F(\eta)d\eta$. The integration contour σ in these representations is chosen in such a way so as not to intersect the singularities of the functions $K^{-1}(\eta)$.

As an example we will consider the problem of the electrical excitation of a piezoelectric crystal layer of thickness h by two strip electrodes; a_{2m-1}, a_{2m} are the beginning and ends of the mth electrode ($m = 1, 2$). We will assume that he surface of the layer is free from mechanical stresses, while in the regions Ω_m: $a_{2m-l} \leqslant x \leqslant a_{2m}$ the electric potentials $f_m = A_m$, $x \in \Omega_m$ are specified. Outside these regions the normal components of the electric induction vector $qm(x) = 0, x \notin \Omega_m$. The lower face of the layer is rigidly clamped, metallized and short-circuited. The system performs steady-state oscillations with frequency ω. The factor $e^{-i\omega t}$, common for all the characteristics, is omitted.

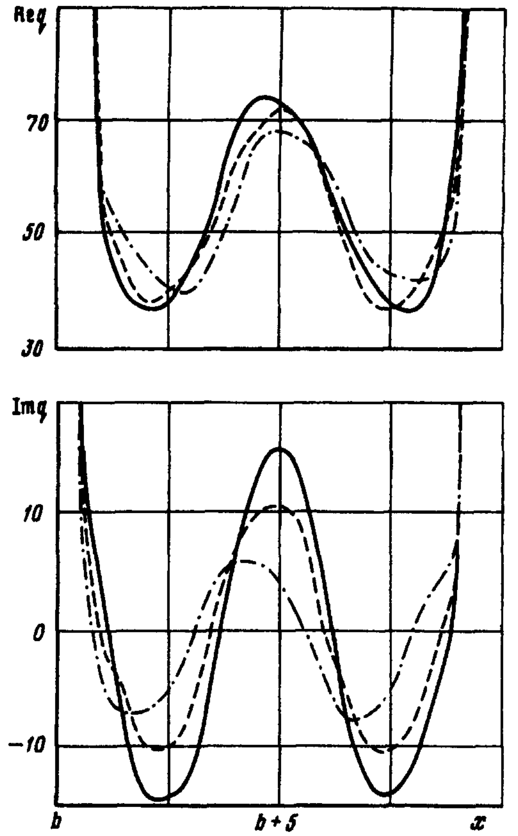

Fig. 1.

This problem can be reduced, by the method of integral transformations [1], to a system of integral equations (1). For XZ-cut piezoelectric crystals of class 6 mm and polarized along the z axis of the piezoelectric ceramic (z is the normal to the surface of the medium) the integrand kernel $K(\alpha)$ possesses the above-mentioned properties 1–3, and was constructed earlier in [5] for different models of the media, namely, a layer, a packet of layers and a multilayered half-space. In this case, the unknown charge density (the electric induction q_m) is found from (14), while the total charges under the electrodes are connected with the Fourier transforms $Q_m(\alpha, \eta)$, described by (16), by the relations

$$G_m = \int_{a_{2m-1}}^{a_{2m}} q_m(x)dx = Q_m(0,0)$$

Figure 1 shows graphs of the real and imaginary parts of the amplitude of the electric induction $q(x) = q_2(x) = -q_1(x)$, referred to c_{44}/L ($L = 10^{10}$ and has the dimensions of electric field) as a function of the distance between the electrodes for $\Omega = 2.6$ and $A_1 = -A_2 = 1$ for U, TC-19 piezoelectric ceramics ($\Omega^2 = \rho\omega^2 c_{44}^{-1} h^2$ is the dimensionless frequency of the oscillations, ρ is the density, c_{44} is the elasticity modulus of the layer, and the parameters $A_1 = 1$ and $A_2 = -1$ correspond to unit electric excitation of the electrodes, here in antiphase). For convenience the graphics of the electric induction are superimposed, apart from the dependence on the value of the separation of the electrodes. The value of the dimensionless parameter $2b$ determines the distance between the electrodes, normalized to the layer thickness. The continuous curve corresponds to $b = 3$, the dashed curve corresponds to $b = 1$ and the dash–dot curve corresponds to $b = 0.25$. The width of both electrodes, normalized to the layer thickness, is equal to 10.

As the distance between the electrode is reduced, the mutual influence of the electrodes increases. When the electrodes are very close, when $A_1 = A_2 = 1$, the electric induction distribution is practically degenerate in the induction distribution under one electrode of double the width, if we eliminate the effect of the singularities at internal points. When the dimensions of the electrodes are increased compared with the distance between them, the mutual influence of the electrodes is reduced.

We wish to thank V. A. Babeshko for his interest and for discussions of the results.

This research was supported financially by the Russian Foundation for Basic Research (96-01-00362) and the "Integratsiya" Purposeful Federal Programme (368).

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Translated by R.C.G.